Note (October 22, 2011).

The present paper is an English summary of the results in the following my master's thesis, which was written in Japanese:

> Masahiro Igarashi, *On generalizations of the sum formula for multiple zeta values*, (Japanese), Master's thesis, Graduate School of Mathematics, Nagoya University, Japan, 2007.

I corrected an error in the above my master's thesis: for the details, see p. 8, lines 6–9 from the top, of the present paper. I also corrected some minor errors in the above my master's thesis. I added some remarks on pp. 10–11 of the present paper.

On generalizations of the sum formula for multiple zeta values

Masahiro Igarashi

Abstract

In the present paper, we prove some generalizations of the sum formula for multiple zeta values by using the method of proving the sum formula for multiple zeta values of Hiroyuki Ochiai.

1 Introduction

The multiple zeta value (MZV for short) is defined by the multiple series

$$\zeta(k_1,\ldots,k_n):=\sum_{0<m_1<\cdots<m_n}\frac{1}{m_1^{k_1}\cdots m_n^{k_n}},$$

where $k_1,\ldots,k_n\in\mathbb{Z}_{\geq 1}$ with $k_n\geq 2$ (Euler [2], Hoffman [4], Zagier [14]; see also [1], [7], [8]). Various relations among MZVs are known. In particular, the sum formula for MZVs is one of the the well-known $\mathbb{Q}$-linear relations among MZVs. The general case of the sum formula for MZVs was first proved by A. Granville [3] and D. Zagier, independently (see also Euler [2], Hoffman [4], Hoffman and Moen [5], Markett [9]). The alternative proofs and the generalizations of the sum formula for MZVs can be found in, e.g., [1], [6], [7], [10], [11], [12].

In the present paper, we prove the following generalizations of the sum formula for MZVs:

Proposition 1. *The identity*

$$\begin{aligned}&\sum_{l=0}^{\infty}\frac{1}{(l+\alpha)^n(l+\beta)^m} \\ &=\sum_{\substack{k_1+\cdots+k_n=m+n-1\\ k_i\in\mathbb{Z}_{\geq 1}}}\sum_{0\leq m_1<\cdots<m_n}\frac{(\alpha)_{m_1}}{m_1!}\frac{m_n!}{(\alpha)_{m_n+1}}\frac{1}{(m_1+\beta)^{k_1}\cdots(m_n+\beta)^{k_n}}\end{aligned} \tag{1}$$

holds for any integers $m,n\geq 1$ *and all* $\alpha,\beta\in\mathbb{C}$ *with* $\operatorname{Re}\alpha>0$, $\operatorname{Re}\beta>0$, *where* $(a)_m$ *denotes the Pochhammer symbol defined by*

$$(a)_m=\begin{cases} a(a+1)\cdots(a+m-1) & \text{if } m\in\mathbb{Z}_{\geq 1},\\ 1 & \text{if } m=0.\end{cases}$$

Taking $\alpha=\beta$ in Proposition 1, we can get the following identity:

Corollary 2. *The identity*

$$\begin{aligned}&\zeta(k;\alpha) \\ &=\sum_{\substack{k_1+\cdots+k_n=k\\ k_i\in\mathbb{Z}_{\geq 1},k_n\geq 2}}\sum_{0\leq m_1<\cdots<m_n}\frac{(\alpha)_{m_1}}{m_1!}\frac{m_n!}{(\alpha)_{m_n}}\frac{1}{(m_1+\alpha)^{k_1}\cdots(m_n+\alpha)^{k_n}}\end{aligned}$$

holds for any integers k,n *such that* $0<n<k$ *and all* $\alpha\in\mathbb{C}$ *with* $\operatorname{Re}\alpha>0$, *where* $(a)_m$ *denotes the Pochhammer symbol and* $\zeta(s;\alpha):=\sum_{m=0}^{\infty}(m+\alpha)^{-s}$ *is the Hurwitz zeta function.*

For the properties of the Hurwitz zeta function, see, e.g., [13].

Taking $\alpha=1$ in Corollary 2, we can get the sum formula for MZVs.

Proposition 3. *The identity*

$$\begin{aligned}&\sum_{\substack{k_1+\cdots+k_n=k\\ k_i\in\mathbb{Z}_{\geq 1},k_n\geq 2}}\zeta(k_1,\ldots,k_n;\alpha) \\ &=\frac{1}{(k-n-1)!}\sum_{l=0}^{\infty}\frac{1}{(l+1)^n}\frac{\partial^{k-n-1}}{\partial X^{k-n-1}}\left\{\frac{(1-X)_l}{(\alpha-X)_{l+1}}\right\}\Bigg|_{X=0}\end{aligned} \tag{2}$$

holds for any integers k, n such that $0 < n < k$ and all $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha > 0$, where $(a)_l$ denotes the Pochhammer symbol and

$$\zeta(s_1, \ldots, s_n; \alpha) := \sum_{0 \le m_1 < \cdots < m_n} \frac{1}{(m_1 + \alpha)^{s_1} \cdots (m_n + \alpha)^{s_n}}$$

is the multiple Hurwitz zeta function.

Taking $\alpha = 1$ in Proposition 3, we can get the sum formula for MZVs.

In order to prove Propositions 1 and 3, we use the method of proving the sum formula for MZVs of Hiroyuki Ochiai. Though Ochiai's proof of the sum formula for MZVs is unpublished, it can be found in [1, pp. 17–20] and [7, pp. 60–61].

2 Proofs of Propositions 1 and 3

In the present section, we prove Propositions 1 and 3 by using the method of proving the sum formula for MZVs of Hiroyuki Ochiai (see [1, pp. 17–20] and [7, pp. 60–61]).

Proof of Proposition 1. Let α and β be complex numbers with positive real parts, and let n be a positive integer. By using the well-known formula

$$\int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

for $a, b \in \mathbb{C}$ with $\operatorname{Re} a > 0$, $\operatorname{Re} b > 0$, we can calculate the generating function of the series on the left-hand side of the identity (1) as follows:

$$\begin{aligned}
&\sum_{m=0}^{\infty} \left(\sum_{l=0}^{\infty} \frac{1}{(l+\alpha)^n (l+\beta)^{m+1}} \right) X^m \\
&= \sum_{l=0}^{\infty} \frac{1}{(l+\alpha)^n (l+\beta-X)} \\
&= \int_0^1 \frac{(1-t_0)^{-X+\beta-1}}{t_0^{\alpha}}\,\mathrm{d}t_0 \int_0^{t_0} \frac{\mathrm{d}t_1}{t_1} \cdots \int_0^{t_{n-2}} \frac{\mathrm{d}t_{n-1}}{t_{n-1}} \int_0^{t_{n-1}} \frac{t_n^{\alpha-1}}{(1-t_n)^{-X+\beta}}\,\mathrm{d}t_n \\
&= \int \cdots \int_{0 < t_n < \cdots < t_0 < 1} \frac{(1-t_0)^{-X+\beta-1} t_n^{\alpha-1}\,\mathrm{d}t_0 \cdots \mathrm{d}t_n}{t_0^{\alpha} t_1 \cdots t_{n-1} (1-t_n)^{-X+\beta}} \qquad (3)
\end{aligned}$$

for all $X \in \mathbb{C}$ such that $|X| < \operatorname{Re}\beta$. Applying the change of variables

$$t_i \longmapsto 1 - t_{n-i}, \quad i = 0, 1, \ldots, n,$$

(see Zagier [14, p. 510]), to the above multiple integral, we get the identity

$$\begin{aligned}
&\int\limits_{0<t_n<\cdots<t_0<1} \cdots \int \frac{(1-t_0)^{-X+\beta-1} t_n^{\alpha-1}\, \mathrm{d}t_0 \cdots \mathrm{d}t_n}{t_0^{\alpha} t_1 \cdots t_{n-1} (1-t_n)^{-X+\beta}} \\
&= \int\limits_{0<t_n<\cdots<t_0<1} \cdots \int \frac{t_n^{-X+\beta-1} (1-t_0)^{\alpha-1}\, \mathrm{d}t_0 \cdots \mathrm{d}t_n}{(1-t_n)^{\alpha} (1-t_{n-1}) \cdots (1-t_1) t_0^{-X+\beta}} \qquad (4)
\end{aligned}$$

for all $X \in \mathbb{C}$ such that $|X| < \operatorname{Re}\beta$. Further, the multiple integral on the

right-hand side of the identity (4) can be calculated as follows:

$$
\begin{aligned}
&\int\!\!\cdots\!\!\int_{0<t_n<\cdots<t_0<1} \frac{t_n^{-X+\beta-1}(1-t_0)^{\alpha-1}\,\mathrm{d}t_0\cdots\mathrm{d}t_n}{(1-t_n)^{\alpha}(1-t_{n-1})\cdots(1-t_1)t_0^{-X+\beta}} \\
&=\int_0^1 \frac{(1-t_0)^{\alpha-1}}{t_0^{-X+\beta}}\,\mathrm{d}t_0 \int_0^{t_0}\frac{\mathrm{d}t_1}{1-t_1}\cdots\int_0^{t_{n-2}}\frac{\mathrm{d}t_{n-1}}{1-t_{n-1}}\int_0^{t_{n-1}}\frac{t_n^{-X+\beta-1}}{(1-t_n)^{\alpha}}\,\mathrm{d}t_n \\
&=\int_0^1 \frac{(1-t_0)^{\alpha-1}}{t_0^{-X+\beta}}\,\mathrm{d}t_0 \int_0^{t_0}\frac{\mathrm{d}t_1}{1-t_1}\cdots\int_0^{t_{n-2}}\frac{\mathrm{d}t_{n-1}}{1-t_{n-1}}\sum_{l_1=0}^{\infty}\frac{(\alpha)_{l_1}}{l_1!}\frac{t_{n-1}^{l_1+\beta-X}}{l_1+\beta-X} \\
&=\cdots\cdots \\
&=\sum_{l_1,\dots,l_n\geq 0}\frac{(\alpha)_{l_1}}{l_1!}\frac{1}{(l_1+\beta-X)\cdots(l_1+\cdots+l_n+n-1+\beta-X)} \\
&\qquad\times\int_0^1(1-t_0)^{\alpha-1}t_0^{l_1+\cdots+l_n+n-1}\,\mathrm{d}t_0 \\
&=\sum_{l_1,\dots,l_n\geq 0}\frac{(\alpha)_{l_1}}{l_1!}\frac{1}{(l_1+\beta-X)\cdots(l_1+\cdots+l_n+n-1+\beta-X)} \\
&\qquad\times\frac{\Gamma(\alpha)\Gamma(l_1+\cdots+l_n+n)}{\Gamma(\alpha+l_1+\cdots+l_n+n)} \\
&=\sum_{0\leq m_1<\cdots<m_n}\frac{(\alpha)_{m_1}}{m_1!}\frac{m_n!}{(\alpha)_{m_n+1}}\frac{1}{(m_1+\beta-X)\cdots(m_n+\beta-X)} \\
&=\sum_{m=0}^{\infty}\left(\sum_{\substack{k_1+\cdots+k_n=m+n\\ k_i\in\mathbb{Z}_{\geq 1}}}\sum_{0\leq m_1<\cdots<m_n}\frac{(\alpha)_{m_1}}{m_1!}\frac{m_n!}{(\alpha)_{m_n+1}}\left\{\prod_{j=1}^{n}\frac{1}{(m_j+\beta)^{k_j}}\right\}\right)X^m
\end{aligned}
$$

for all $X\in\mathbb{C}$ such that $|X|<\operatorname{Re}\beta$. By combining this result and the identities (3) and (4), the generating functions of both sides of the identity (1) coincide. This completes the proof of Proposition 1. $\square$

Proof of Proposition 3. Let α be a complex number with positive real part, and let n be a positive integer. By the same calculation as in the proof of Proposition 1, we can get the following multiple integral representation of

the generating function of the sums on the left-hand side of the identity (2):

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\left(\sum_{\substack{k_1+\cdots+k_n=m+n+1\\ k_i\in\mathbb{Z}_{\geq 1},k_n\geq 2}}\zeta(k_1,\ldots,k_n;\alpha)\right)X^m\\
&=\int\cdots\int_{0<t_n<\cdots<t_0<1}\frac{t_0^{X-1}t_n^{\alpha-X-1}\,\mathrm{d}t_0\cdots\mathrm{d}t_n}{(1-t_1)\cdots(1-t_n)} \qquad (5)
\end{aligned}
$$

for all $X\in\mathbb{C}$ such that $|X|<\operatorname{Re}\alpha$. Applying the change of variables

$$
t_i \longmapsto 1-t_{n-i}, \quad i=0,1,\ldots,n,
$$

(see Zagier [14, p. 510]), to the above multiple integral, we get the identity

$$
\begin{aligned}
&\int\cdots\int_{0<t_n<\cdots<t_0<1}\frac{t_0^{X-1}t_n^{\alpha-X-1}\,\mathrm{d}t_0\cdots\mathrm{d}t_n}{(1-t_1)\cdots(1-t_n)}\\
&=\int\cdots\int_{0<t_n<\cdots<t_0<1}\frac{(1-t_0)^{\alpha-X-1}\,\mathrm{d}t_0\cdots\mathrm{d}t_n}{(1-t_n)^{1-X}t_{n-1}\cdots t_0} \qquad (6)
\end{aligned}
$$

for all $X\in\mathbb{C}$ such that $|X|<\operatorname{Re}\alpha$. Further, by the same calculation as in the proof of Proposition 1, the multiple integral on the right-hand side of the identity (6) can be expressed as

$$
\begin{aligned}
&\int\cdots\int_{0<t_n<\cdots<t_0<1}\frac{(1-t_0)^{\alpha-X-1}\,\mathrm{d}t_0\cdots\mathrm{d}t_n}{(1-t_n)^{1-X}t_{n-1}\cdots t_0}\\
&=\sum_{l=0}^{\infty}\frac{(1-X)_l}{(\alpha-X)_{l+1}}\frac{1}{(l+1)^n} \qquad (7)
\end{aligned}
$$

for all $X\in\mathbb{C}$ such that $|X|<\operatorname{Re}\alpha$. By using the expansion

$$
\frac{(1-X)_l}{(\alpha-X)_{l+1}}=\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m}{\partial X^m}\left\{\frac{(1-X)_l}{(\alpha-X)_{l+1}}\right\}\Bigg|_{X=0}X^m, \quad |X|<\operatorname{Re}\alpha,
$$

the series on the right-hand side of the identity (7) can be written as

$$\begin{aligned}
&\sum_{l=0}^{\infty} \frac{(1-X)_l}{(\alpha-X)_{l+1}} \frac{1}{(l+1)^n} \\
&= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m}{\partial X^m} \left\{ \frac{(1-X)_l}{(\alpha-X)_{l+1}} \right\} \Bigg|_{X=0} \frac{X^m}{(l+1)^n}
\end{aligned} \tag{8}$$

for all $X \in \mathbb{C}$ such that $|X| < \operatorname{Re}\alpha$. We note that the double series on the right-hand side of the identity (8) converges absolutely for $X \in \mathbb{C}$ such that $|X| < r$, where r is a fixed real number such that $0 < r < \operatorname{Re}\alpha/2$. (In my master's thesis, to prove the absolute convergence of the above double series, I imposed the condition $|X| < \operatorname{Re}\alpha/2$ instead of the condition $|X| < r$, $(0 < r < \operatorname{Re}\alpha/2)$. I think that the condition $|X| < \operatorname{Re}\alpha/2$ is incorrect.) Therefore, we get the identity

$$\begin{aligned}
&\int \cdots \int_{0<t_n<\cdots<t_0<1} \frac{(1-t_0)^{\alpha-X-1}\,\mathrm{d}t_0 \cdots \mathrm{d}t_n}{(1-t_n)^{1-X} t_{n-1} \cdots t_0} \\
&= \sum_{m=0}^{\infty} \left(\sum_{l=0}^{\infty} \frac{1}{(l+1)^n} \frac{\partial^m}{\partial X^m} \left\{ \frac{(1-X)_l}{(\alpha-X)_{l+1}} \right\} \Bigg|_{X=0} \right) \frac{X^m}{m!}
\end{aligned}$$

for all $X \in \mathbb{C}$ such that $|X| < r$. By combining this identity and the identities (5) and (6), the generating functions of both sides of the identity (2) coincide. This completes the proof of Proposition 3. $\square$

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Remarks (October 22, 2011).

(i) Ochiai's proof of the sum formula for MZVs can also be found on pp. 19–22 of the following lecture note:

> T. Arakawa and M. Kaneko, *Introduction to multiple zeta values*, (Japanese), Kyushu University COE Lecture Note Series, Vol. 23 (2010).

(ii) The present research depends on an unpublished work of Hiroyuki Ochiai, i.e., Ochiai's proof of the sum formula for MZVs. In particular, I found the multiple series in Proposition 1 and Corollary 2 by studying Ochiai's proof of the sum formula for MZVs. (I think that this matter can be confirmed by examining the multiple integral representation of the generating function of sums of MZVs in Ochiai's proof of the sum formula for MZVs (see [1, pp. 17–20], [7, pp. 60–61], and the above lecture note of T. Arakawa and M. Kaneko).) I think that, in February 2007, several people were already aware of the content of my master's thesis.

(iii) In the paper arXiv:0908.2536v7, I cited several prior works of other researchers related to the present research.

(iv) Multiple series similar to those in Proposition 1 and Corollary 2 appear in the study of the ε-expansion of the hypergeometric function; see, e.g., the following papers:

> A. I. Davydychev and M. Yu. Kalmykov, *Massive Feynman diagrams and inverse binomial sums*, Nucl. Phys. B **699** (2004), 3–64.

> M. Yu. Kalmykov, *Series and ε-expansion of the hypergeometric functions*, Nucl. Phys. B (Proc. Suppl.) **135** (2004), 280–284.

> M. Yu. Kalmykov and B. A. Kniehl, *Towards all-order Laurent expansion of generalised hypergeometric functions about rational values of parameters*, Nucl. Phys. B **809** [PM] (2009), 365–405.

> M. Yu. Kalmykov and O. Veretin, *Single-scale diagrams and multiple binomial sums*, Phys. Lett. B **483** (2000), 315–323.

M. Yu. Kalmykov, B. F. L. Ward and S. A. Yost, *Multiple (inverse) binomial sums of arbitrary weight and depth and the all-order ε-expansion of generalized hypergeometric functions with one half-integer value of parameter*, J. High Energy Phys. 10 (2007) 048.

For multiple series similar to those in Proposition 1 and Corollary 2, see also the paper arXiv:0908.2536v7.